

## Binary Response Models

### 1 Introduction

Many dependent variables of interest in economics and other social sciences can only take two values. The two possible outcomes are usually denoted by 0 and 1. Such variables are called *dummy* variables or *dichotomous* variables. Some examples:

- The labor market status of a person. The variable takes the value 1 if a person is employed and 0 if he is unemployed. The values 1 and 0 can be assigned arbitrarily.
- Voting behavior of a person. The variable takes 1 if the person votes in favor of a new policy and 0 otherwise. Again the values 1 and 0 are arbitrary.

The expected value of a dichotomous variable  $y_i \in \{0, 1\}$  is the probability that it takes the value 1:

$$E(y_i) = 0 \cdot P(y_i = 0) + 1 \cdot P(y_i = 1) = P(y_i = 1).$$

The linear regression model, called *linear probability model* in this context,

$$y_i = x_i' \beta + u_i, \quad E(u_i | x_i) = 0$$

is therefore not an adequate statistical model as the expected value  $E(y_i | x_i) = x_i' \beta$  can lie outside  $[0, 1]$  and does not represent a probability. In addition, the error term is heteroscedastic as  $V(u_i | x_i) = x_i' \beta (1 - x_i' \beta)$  depends on  $x_i$ .

Note: Despite this logical inconsistency, OLS with robust standard errors is in practice an adequate method to estimate average marginal effects in binary choice models.

### 2 The Econometric Model: Probit and Logit

Binary response models directly describe the response probabilities  $P(y_i = 1)$  of the dependent variable  $y_i$  as a function of explanatory variables  $x_i'$ .

Consider a sample of  $N$  independently and identically distributed (i.i.d.) observations  $i = 1, \dots, N$  of the dependent dummy variable  $y_i$  and a  $(K + 1)$ -dimensional row vector  $x_i'$  of explanatory variables. The probability that the dependent variable takes value 1 as modeled as

$$P(y_i = 1 | x_i) = F(x_i' \beta)$$

where  $\beta$  is a  $(K + 1)$ -dimensional column vector of parameters and  $x_i' \beta$  is a *single linear index*. The *transformation function*  $F$  maps the single index into  $[0, 1]$  and satisfies in general

$$F(-\infty) = 0, \quad F(\infty) = 1, \quad \partial F(z) / \partial z > 0.$$

The *probit* model assumes that the transformation function  $F$  is the cumulative density function (cdf) of the standard normal distribution. The response probabilities are then

$$P(y_i = 1 | x_i) = \Phi(x_i' \beta) = \int_{-\infty}^{x_i' \beta} \phi(z) dz$$

where  $\phi(\cdot)$  is the pdf and  $\Phi(\cdot)$  the cdf of the standard normal distribution.

In the *logit* model, the transformation function  $F$  is the logistic function. The response probabilities are then

$$P(y_i = 1 | x_i) = \Lambda(x_i' \beta) = \frac{e^{x_i' \beta}}{1 + e^{x_i' \beta}} = \frac{1}{1 + e^{-x_i' \beta}}$$

Figure 1 shows the transformation function  $F$  for the probit and the logit model.

Note: The Logit and Probit model are almost identical and the choice of the model is usually arbitrary. However, the parameters  $\beta$  of the two

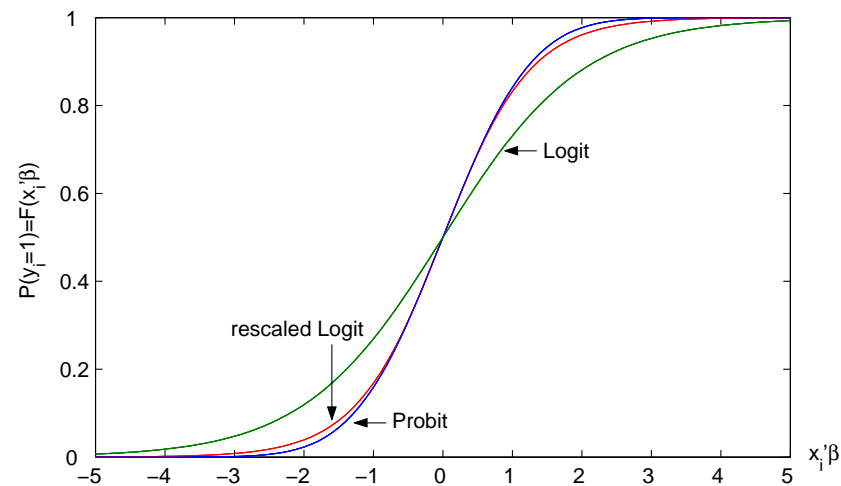


Figure 1: The transformation function in the probit and logit model.

models are scaled differently. Multiplying the parameters in the logit model by 1.6 are approximately the same as the probit estimates.<sup>1</sup>

### 3 Latent Variable Model

There is an alternative interpretation that gives rise to the probit (and analogously the logit) model. Consider a *latent* variable which is not observed by the researcher and linearly depends on  $x_i$

$$y_i^* = x_i'\beta + u_i, \quad E(u_i) = 0$$

The latent variable  $y_i^*$  can be interpreted as the utility difference between choosing  $y_i = 1$  and 0. In this case, the model is called a *random utility*

<sup>1</sup>The factor 1.6 equals the first derivative of  $F$  at  $x_i'\beta = 0$  and is in most applications the appropriate rescaling. A different approach is to equal the variance of the distribution for which  $F$  is the cdf. For the probit model the variance is 1 and for the logit model  $\pi/\sqrt{3} \cong 1.81$ .

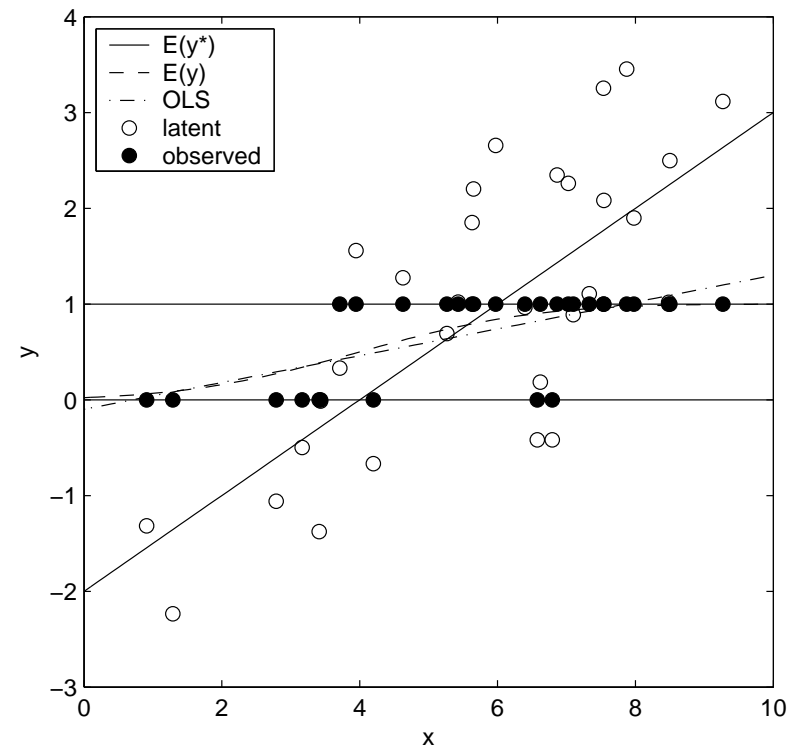


Figure 2: The probit model with a latent variable.  $N = 30$ ,  $K = 2$  (a constant and one independent variable) and  $\beta = (-2, 0.5)$ .

model.

Only the choice  $y_i$  is observed by the researcher. An individual chooses  $y_i = 1$  if the latent variable is positive and 0 otherwise, hence the observed variable is

$$y_i = \begin{cases} 1 & \text{if } y_i^* > 0 \\ 0 & \text{if } y_i^* \leq 0 \end{cases}$$

Furthermore, assume that the individual observations  $(x_i, y_i)$  are i.i.d.,

that the explanatory variables are *exogenous* and that the error term is normally distributed

$$u_i|x_i \sim N(0, \sigma^2)$$

The probability that individual  $i$  chooses  $y_i = 1$  can now be derived from the latent variable and the decision rule, i.e.

$$\begin{aligned} P(y_i = 1|x_i) &= P(y_i^* > 0|x_i) = P(x_i'\beta + u_i > 0|x_i) = P(u_i > -x_i'\beta|x_i) \\ &= 1 - \Phi(-x_i'\beta/\sigma) = \Phi(x_i'\beta/\sigma). \end{aligned}$$

The probit model arises when  $\sigma^2$  is set to unity.

Note:  $\beta$  and  $\sigma$  are *not separately identified* as only the ratio  $\beta/\sigma$  can be estimated. Figure 2 visualizes the latent variable model.

## 4 Interpretation of the Parameters

Different from the linear regression model, the parameters  $\beta$  cannot directly be interpreted as marginal effects on the dependent variable  $y_i$ . In some situations, the index function  $x_i'\beta$  has a clear interpretation in a theoretical model and the marginal effect  $\beta_k$  of a change in the independent variable  $x_{ik}$  on  $y_i^*$  is meaningful. Even then, the marginal effect is only identified if there is reason to set  $\sigma^2$  to unity.

In general, we are interested in the marginal effect of a change in  $x_{ik}$  on the expected value of the observed variable  $y_i$ , i.e.

$$\text{Probit: } \frac{\partial E(y_i|x_i)}{\partial x_{ik}} = \frac{\partial P(y_i = 1|x_i)}{\partial x_{ik}} = \phi(x_i'\beta) \beta_k$$

$$\text{Logit: } \frac{\partial E(y_i|x_i)}{\partial x_{ik}} = \frac{\partial P(y_i = 1|x_i)}{\partial x_{ik}} = \Lambda(x_i'\beta) [1 - \Lambda(x_i'\beta)] \beta_k$$

This marginal effect depends on the characteristics  $x_i$  of observation  $i$ . Therefore, any individual has a different marginal effect. There are several ways to summarize and report the information in the model. A first possibility is to present the marginal effects for the “mean type” ( $x_i = \bar{x}_i$ ),

the “median type”, or some interesting extreme types. A second approach is to calculate the marginal effects for all observations in the sample and report the mean of the effects.

The estimated model can also be used for predictions

$$\text{Probit: } \hat{P}(y_i = 1|x_i) = \Phi(x_i'\hat{\beta})$$

$$\text{Logit: } \hat{P}(y_i = 1|x_i) = \Lambda(x_i'\hat{\beta})$$

This information can be aggregated to, for example, the proportion of observations with  $y_i = 1$ . There are two prediction methods for the aggregate proportion: (1) assume  $\hat{y}_i = 1$  if  $\hat{P}_i > 0.5$  and calculate  $\bar{\hat{y}} = 1/N \sum_i \hat{y}_i$ . (2) Sum the predicted choice probabilities calculate  $\hat{P} = \sum_i \hat{P}(y_i = 1|x_i)$ . The two measures can be contrasted to the actual fraction. Method 1 also allows to compare actual and predicted outcomes for any observation. It is also often interesting to report and contrast predicted probabilities for certain types of individuals.

## 5 Estimation with Maximum Likelihood

The probit and logit models are estimated by maximum likelihood (ML). Assuming independence across observations, the likelihood function is

$$\begin{aligned} \mathcal{L} &= \prod_{\{i|y_i=0\}} P(y_i = 0|x_i) \prod_{\{i|y_i=1\}} P(y_i = 1|x_i) \\ &= \prod_{i=1}^N [1 - F(x_i'\beta)]^{1-y_i} F(x_i'\beta)^{y_i} \end{aligned}$$

where  $P(y_i = 1|x_i) = F(x_i'\beta) = \Phi(x_i'\beta)$  in the probit model and  $P(y_i = 1|x_i) = F(x_i'\beta) = \Lambda(x_i'\beta)$  in the logit model. The corresponding log likelihood function is

$$\log \mathcal{L} = \sum_{i=1}^N [y_i \log F(x_i'\beta) + (1 - y_i) \log (1 - F(x_i'\beta))]$$

The first order conditions for an optimum are in general

$$\frac{\partial \log \mathcal{L}}{\partial \beta} = \sum_{i=1}^N \left[ y_i \frac{f(x'_i \beta)}{F(x'_i \beta)} + (1 - y_i) \frac{-f(x'_i \beta)}{1 - F(x'_i \beta)} \right] x'_i = 0$$

where  $f(z) \equiv \partial F(z)/\partial z$ . This simplifies in the probit model to

$$\frac{\partial \log \mathcal{L}}{\partial \beta} = \sum_{\{i|y_i=0\}} \frac{-\phi(x'_i \beta)}{1 - \Phi(x'_i \beta)} x'_i + \sum_{\{i|y_i=1\}} \frac{\phi(x'_i \beta)}{\Phi(x'_i \beta)} x'_i = 0$$

and in the logit model to

$$\frac{\partial \log \mathcal{L}}{\partial \beta} = \sum_{i=1}^N (y_i - \Lambda(x'_i \beta)) x'_i = 0.$$

There is no analytical solution to these FOCs and numerical optimization routines are used. The log likelihood function can be shown to be globally concave for both models and numerical routines converge well to the unique global maximum.

The ML estimator of  $\beta$  is consistent and asymptotically normally distributed. The approximate distribution in large samples is

$$\hat{\beta} \stackrel{A}{\sim} N(\beta, I(\beta)^{-1})$$

where the inverse information matrix  $I^{-1}$  can be estimated by one of the standard ML procedures (inverse expected hessian matrix, inverse hessian matrix, BHHH, or Eicker-Huber-White-Sandwich). Asymptotic hypothesis tests are performed as Wald, likelihood ratio or lagrange multiplier tests.

The ML estimation of the probit model (and analogously the logit model) rests on the strong assumption that the latent error term is normally distributed and homoscedastic. The ML estimator is inconsistent in the presence of heteroscedasticity and robust covariance estimators cannot solve this. Several *semi-parametric* estimation strategies have been proposed that relax the distributional assumption about the error term. See Horowitz and Savin (2001) for an introduction and Gerfin (1996) for a nice comparison of different estimators.

## 6 Implementation in Stata 10.0

Stata estimates the probit model by the command

```
probit depvar [indepvars]
```

and the logit model by

```
logit depvar [indepvars]
```

where *depvar* is the dependent dummy variable and *indepvars* is a list of explanatory variables. For example,

```
webuse auto.dta
probit foreign weight mpg
```

Stata reports the inverse hessian matrix as default covariance estimator. The option `vce(opg)` reports BHHH and option `vce(robust)` the sandwich covariance estimator.

Response probabilities are estimated for each observation with the post-estimation command `predict`. For example,

```
predict p_foreign, pr
```

predicts the probability of being foreign for each car model. Marginal effects for specific types are calculated with the post-estimation command `mfx`. For example,

```
mfx
```

calculates the marginal effects for the mean type, e.g. a car with average weight and mpg. The marginal effects for a specific type, e.g. a car with weight of 2000 lbs. and 40 mpg is reported by

```
mfx, at(weight = 2000, mpg = 40)
```

In the case of explanatory dummy variables, Stata reports the exact discrete effect. Marginal effects for each individual in the sample can be calculated manually. For example in the probit model,

```
probit foreign weight mpg
predict xb, xb
predictnl mfx_weight = normalden(xb)*_b[weight]
```

or in the logit model

```
logit foreign weight mpg
predict p, pr
predictnl mfx_weight = p*(1-p)*_b[weight]
```

The post-estimation command `predictnl` cannot be used to calculate confidence intervals for individual marginal effects as the error as prediction error in  $p$  and  $xb$ , respectively, is ignored.

## References

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